TRINITY COLLEGE

Mathematics Specialist Units 1,2
Test 6 2017

Section 1 Calculator Free
Complex Numbers, Proof

STUDENT'S NAME ______________________________

DATE: Thursday 7 September **TIME:** 50 minutes **MARKS**: 58

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (5 marks)

Express the following recurring decimals as a fraction.

(a) $0.\overline{123}$ [2]

$x = 0.\overline{123}$

$1000x = 123.\overline{123}$

$999x = 123$

$x = \frac{123}{999}$

(b) $6.8\overline{07}$ [3]

$x = 6.8\overline{07}$

$10x = 68.\overline{07}$

$1000x = 6807.\overline{07}$

$990x = 6739$

$x = \frac{6739}{990}$

2. (3 marks)

Prove the product of three consecutive even whole numbers is a multiple of 8.

$$(2x-2)2x(2x+2) = 2x(4x^2-4)$$
$$= 8x^3 - 8x$$
$$= 8(x^3-x)$$

$\therefore$ MULTIPLE OF 8

3. (6 marks)

(a) Prove $\sqrt{11}$ is irrational. [3]

ASSUME $\sqrt{11}$ RATIONAL

(a, b CO PRIME)

$$\sqrt{11} = \frac{a}{b}$$
$$11 = \frac{a^2}{b^2}$$
$$11b^2 = a^2$$

$\Rightarrow a, b$ BOTH ODD (CAN'T BE BOTH EVEN)

$\Rightarrow a^2$ MULTIPLE OF 11

SINCE 11 PRIME, a ALSO MULTIPLE OF 11

$\Rightarrow a = 11k$

$11b^2 = (11k)^2$

$11b^2 = 11^2k^2$

$b^2 = 11k^2$

$\Rightarrow b$ A MULTIPLE OF 11

BUT NOT POSSIBLE SINCE a, b CO PRIME

$\therefore$ ASSUME NOT TRUE

ie $\sqrt{11}$ IRRATIONAL

(b) Prove $\log_3 7$ is irrational. [3]

ASSUME $\log_3 7$ RATIONAL

$$\log_3 7 = \frac{a}{b}$$

a, b CO PRIME

$$7 = 3^{\frac{a}{b}}$$
$$7^b = 3^a$$

NOT POSSIBLE FOR INTEGER a, b

$\therefore$ CONTRADICTS ASSUMPTION

$\therefore$ $\log_3 7$ RATIONAL

4. (6 marks)

Solve

(a) $2x^2 + 3x + 7 = 0$ [3]

$$x = \frac{-3 \pm \sqrt{9-56}}{4}$$

$$= \frac{-3 \pm \sqrt{-47}}{4}$$

$$= \frac{-3 \pm i\sqrt{47}}{4}$$

(b) $z - 2\bar{z} = 4 + 3i$ (Hint: let $z = a + bi$) [3]

$$a + bi - 2(a - bi) = 4 + 3i$$

$$-a + 3bi = 4 + 3i$$

$$a = -4$$

$$b = 1$$

5. (23 marks)

Given $w = 5 - 4i$ and $z = -2 + 3i$

(a) Determine

(i) z^2 [2]

$(-2+3i)(-2+3i)$

$= 4 - 6i - 6i + 9i^2$

$= -5 - 12i$

(ii) $w\bar{z}$ [2]

$(5-4i)(-2-3i)$

$= -10 - 15i + 8i + 12i^2$

$= -22 - 7i$

(iii) $\dfrac{w}{z}$ [3]

$\dfrac{5-4i}{-2+3i} \times \dfrac{-2-3i}{-2-3i}$

$= \dfrac{-22-7i}{13}$

(iv) $\mathrm{Im}(w+iz)$ [2]

$Im\left(5-4i + i(-2+3i)\right)$

$= Im(2-6i)$

$= -6$

(b) Determine whether $\bar{w} \times \bar{z} = \left(\overline{wz}\right)$ [4]

$(5+4i)(-2-3i)$

$= -10 - 15i - 8i - 12i^2$

$= 2 - 23i$

$\overline{(5-4i)(-2+3i)}$

$= \overline{(-10+15i+8i-12i^2)}$

$= \overline{(2+23i)}$

$= 2 - 23i$

YES

(c) Determine a and b if $a + bi = i\left(w^{-1}\right)$ [4]

$a+bi = \dfrac{i}{5-4i} \times \dfrac{5+4i}{5+4i}$

$a+bi = \dfrac{-4+5i}{25+16}$

$a+bi = \dfrac{-4}{41} + \dfrac{5i}{41}$

$a = -\dfrac{4}{41} \qquad b = \dfrac{5}{41}$

(d) Locate each of the following on the Argrand Plane shown.

(i) $-2i$ [1]

(ii) iw [2]

$i(5-4i)$

$= 5i + 4$

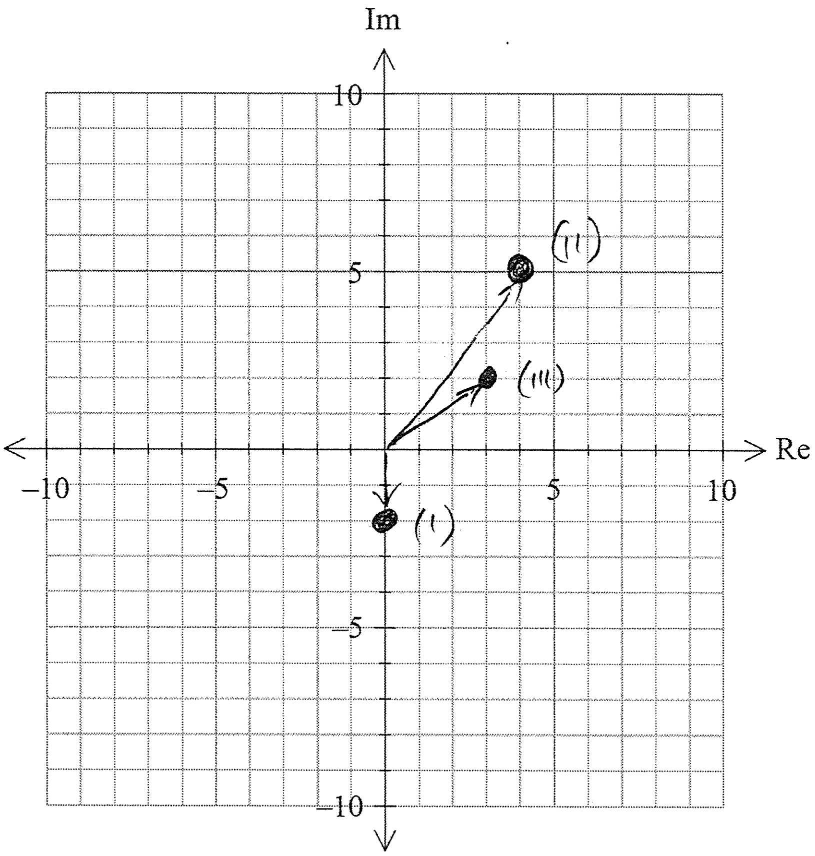
(iii) $\frac{z}{i}$ [3]

$\frac{-2}{i} + \frac{3i}{i}$

$= \frac{-2i^4}{i} + 3$

$= -2i^3 + 3$

$= 2i + 3$



6. (12 marks)

Use proof by induction for each of the following.

(a) Prove $3^{2n}-1$ is divisible by 8 for integer $n \geq 1$. [6]

$n=1$ $\quad 3^2 - 1 = 8$

$\therefore$ DIVISIBLE BY 8

ASSUME TRUE FOR $n = k$

IE $3^{2k} - 1$ DIVISIBLE BY 8

PROVE TRUE FOR $n = k+1$

IE PROVE $3^{2(k+1)} - 1$ DIVISIBLE BY 8

$$3^{2k+2} - 1$$

$$= 3^2 \times 3^{2k} - 1$$

$$= 9 \times 3^{2k} - 9 + 8$$

$$= 9(3^{2k} - 1) + 8$$

DIVISIBLE BY 8 $\qquad$ DIVISIBLE BY 8

$\Rightarrow$ TRUE FOR $n = k+1$

SINCE TRUE FOR $n=1$ THEN TRUE FOR $n=2$

SINCE TRUE FOR $n=2$ THEN TRUE FOR $n=3$

AND SO ON

$\therefore$ $3^{2n} - 1$ DIVISIBLE BY 8 FOR INTEGER $n \geq 1$

(b) Prove $\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}^n = \begin{bmatrix} \cos nx & -\sin nx \\ \sin nx & \cos nx \end{bmatrix}$ [6]

LET $n=1$

$LHS = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$ $\quad RHS = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$

$\therefore$ TRUE FOR $n=1$

ASSUME TRUE FOR $n=k$

ie $\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}^k = \begin{bmatrix} \cos kx & -\sin kx \\ \sin kx & \cos kx \end{bmatrix}$

PROVE TRUE FOR $n=k+1$

ie PROVE $\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}^{k+1} = \begin{bmatrix} \cos (k+1)x & -\sin (k+1)x \\ \sin (k+1)x & \cos (k+1)x \end{bmatrix}$

$LHS = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}^k \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$

$= \begin{bmatrix} \cos kx & -\sin kx \\ \sin kx & \cos kx \end{bmatrix} \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$

$= \begin{bmatrix} \cos x \cos kx - \sin x \sin kx & -\sin x \cos kx - \sin kx \cos x \\ \sin kx \cos x + \sin x \cos kx & -\sin x \sin kx + \cos x \cos kx \end{bmatrix}$

$= \begin{bmatrix} \cos (x + kx) & -\sin (kx + x) \\ \sin (kx + x) & \cos (kx + x) \end{bmatrix}$

$= \begin{bmatrix} \cos (k+1)x & -\sin (k+1)x \\ \sin (k+1)x & \cos (k+1)x \end{bmatrix}$

$= RHS$

$\Rightarrow$ TRUE FOR $n=k+1$ WHEN TRUE FOR $n=k$

SINCE TRUE FOR $n=1$ MUST BE TRUE FOR $n=2$

SINCE TRUE FOR $n=2$ MUST BE TRUE FOR $n=3$

AND SO ON

$\therefore$ TRUE FOR ALL INTEGER $n \geq 1$

7. (3 marks)

Prove every prime number greater than 4 is either one more or one less than a multiple of 6.

$6n + 0$	DIVISIBLE BY 6	NOT PRIME
$6n + 1$	COULD BE PRIME	
$6n + 2$	DIVISIBLE BY 2	NOT PRIME
$6n + 3$	DIVISIBLE BY 3	NOT PRIME
$6n + 4$	DIVISIBLE BY 2	NOT PRIME
$6n + 5$	COULD BE PRIME	

$\therefore$ ONLY ONE MORE OR ONE LESS THAN A MULTIPLE OF 6 ARE POSSIBLE PRIMES.